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Analytic Solution of the Riccati Equation for the Homing Missile Linear-Quadratic Control Problem

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Introduction

THIS Note presents an analytic solution of the matrix Riccati differential equation with a terminal boundary condition for a special case. Assuming a solution of the form $S(t) = P(t)/f(t)$, we obtain linear differential equations for determining the matrix $P(t)$ and the scalar function $f(t)$. It is well known that in the linear quadratic optimal control problem for homing systems, control sequences that minimize a given performance index may be calculated in terms of solutions to the matrix Riccati equation.[1-4] Many effective methods exist for solving the matrix Riccati equation. However, this Note presents a simple analytic means of solving the matrix Riccati differential equation of the linear-quadratic control problem for homing systems. Representation formulas are given for the general solution of the matrix Riccati differential equation

$$\dot{S} + SF + F^TS - SGR^{-1}G^TS = 0$$

using $S = P(t)/f(t)$, where $f(t)$ and $P(t)$ are solutions of first-order ordinary linear differential equations. The given technique is restricted to single input.

General Problem and Solution

The most general problem being considered here is to minimize the quadratic performance index

$$J = \frac{1}{2}(x^TS_fx)_{t_f} + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + u^TRu)\,\mathrm{d}t \tag{1}$$

subject to the linear homogeneous differential equations of constraint

$$\dot{x} = F(t)x + G(t)u \tag{2}$$

and the zero terminal miss distance boundary constraint

$$x(t_f) \simeq 0 \tag{3}$$

where x is the n-component state vector, u is the m-component control vector, S_f and Q are positive semidefinite matrices, R is a positive definite matrix, and t_f is terminal time. It is assumed that the initial state $x(t_0)$ is given.

The necessary conditions for optimality are

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x}; \qquad \lambda(t_f) = S_fx(t_f) \tag{4}$$

$$0 = \frac{\partial H}{\partial u} \tag{5}$$

where

$$H = \tfrac{1}{2}x^TQx + \tfrac{1}{2}u^TRu + \lambda^T(Fx + Gu) \tag{6}$$

Substituting Eq. (6) into Eq. (5), the solution for the optimal closed-loop control law is obtained.

$$u = -R^{-1}G^T\lambda \tag{7}$$

Now substitution of Eq. (7) into Eqs. (2) and (4) gives

$$\begin{vmatrix} \dot{x} \\ \dot{\lambda} \end{vmatrix} = \begin{vmatrix} F & -GR^{-1}G^T \\ -Q & -F^T \end{vmatrix} \begin{vmatrix} x \\ \lambda \end{vmatrix} \tag{8}$$

with

$$x(t_0) = x_0 \qquad \text{and} \qquad \lambda(t_f) = S_fx_f$$

We may assume a solution for Eq. (8) of the form

$$\lambda(t) = S(t)x(t) \tag{9}$$

then

$$\dot{\lambda} = \dot{S}(t)x(t) + S(t)\dot{x}(t) \tag{10}$$

Substituting Eqs. (8) and (9) into Eq. (10), we have

$$\dot{S} + SF + F^TS - SGR^{-1}G^TS + Q = 0 \tag{11}$$

with boundary condition

$$S_f = S(t_f) \tag{12}$$

For the special case, which is considered in the paper by Cottrell,[1] $Q = 0$; Eq. (11) becomes

$$\dot{S} + SF + F^TS - SGR^{-1}G^TS = 0 \tag{13}$$

Since S_f is symmetric and Eqs. (11) or (13) are symmetric, it is clear that S is a symmetric matrix. In this Note, we show one way to solve Eq. (13) with the boundary condition

$$S_f = \begin{vmatrix} C_1 & 0\cdots0 \\ 0 & 0\cdots0 \\ \cdot & \cdots\cdots \\ \cdot & \cdots\cdots \\ \cdot & \cdots\cdots \\ 0 & 0\cdots0 \end{vmatrix} \tag{14}$$

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and with the special case of the matrix

$$G^T = [0, \ldots, 0, g_r, 0, \ldots, 0] \tag{15}$$

Assume that there exists a symmetric matrix

$$S(t) = \frac{1}{f(t)} P(t) \tag{16}$$

which is the solution of the matrix Riccati equation (13) with the boundary condition (14), where $f(t)$ is a scalar function and $P(t)$ is a matrix function. According to Eq. (16), relations (14) and (12) for $t = t_f$ we have

$$P(t_f) = \left| \begin{array}{ll} 1 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \cdot & \cdots\cdots \\ \cdot & \cdots \\ \cdot & \cdots\cdots \\ 0 & 0 \cdots 0 \end{array} \right|, \qquad f(t_f) = \frac{1}{C_1} \tag{17}$$

Substituting $S(t)$ from Eq. (16) into Eq. (13) yields

$$-\frac{f'}{f^2} P + \frac{1}{f}(\dot{P} + PF + F^T P) - \frac{1}{f^2} PGR^{-1} G^T P = 0 \tag{18}$$

Using the fact that $S(t)$ is a symmetric matrix we see that the matrix $P(t)$ must be a symmetric matrix. We now assume that the matrix $P(t)$ can be written in the form

$$P = DD^T \tag{19}$$

where

$$D^T = [d_1, d_2, \ldots, d_i, \ldots, d_n] \tag{20}$$

and d_i is a scalar function. This restricts P and therefore $S(t)$ is of rank one, which is permissible since it is consistent with Eq. (14). Combining Eqs. (18) and (19), we have

$$-\frac{f'}{f^2} DD^T + \frac{1}{f} [(\dot{D} + F^T D) D^T + D(\dot{D}^T + D^T F)]$$

$$-\frac{1}{f^2} DD^T GR^{-1} G^T DD^T = 0 \tag{21}$$

Equation (21) is satisfied for those functions $f(t)$ and $D(t)$ that are determined by the following equations:

$$\dot{D} + F^T D = 0 \tag{22}$$

$$f' DD^T = -DD^T GR^{-1} G^T DD^T \tag{23}$$

with the boundary condition

$$D^T(t_f) = [1, 0, \ldots, 0], \qquad f(t_f) = \frac{1}{C_1} \tag{24}$$

Equation (22) corresponds to the second equation of system (8) for $Q = 0$. In this way, we show that the problem of solving the matrix Riccati equation (13) with the boundary condition (14) is reduced to solving a suitable set of linear differential equations (22) and (23) with the boundary condition (24).

Now, we will solve the linear differential equations (22) and (23) for the case where $R = a$ (constant parameter) and the matrix G is given as

$$G^T = [0, \ldots, 0, g_r, 0, \ldots, 0] \tag{25}$$

Since we have assumed that the matrix $P = DD^T$ is a singular matrix, it is not invertible. Thus, from the definition of the product of matrices and the equality of matrices, we can solve Eq. (23); that is

$$f' = -\frac{1}{a} g_r^2 d_r^2 \tag{26}$$

Solving Eq. (26), we can find the unknown function as

$$f(t) \Big|_{f(t_0)}^{1/c_1} = -\frac{g_r^2}{a} \int_{t_0}^{t_f} d_r^2 \, \mathrm{d}t \tag{27}$$

Using formulas (7), (9), (16), and (19), it is possible to write the expression for the optimal closed-loop control law, that is

$$u = -\frac{1}{f(t)} R^{-1} G^T DD^T \mathbf{x} \tag{28}$$

Now, using Eqs. (28), (20), and (15) for $R = a$, we have

$$u = -\frac{1}{af(t)} g_r d_r M \tag{29}$$

where

$$M = D^T \mathbf{x} \tag{30}$$

Special Case for the Homing System

The intercept geometry is shown in Fig. 1, where the following variables are defined: φ is the line of sight angle, y is the relative position, a_{Tn} is the target normal acceleration, a_{Mn} is the missile normal acceleration, and r is the relative range vector. The dynamics are the same as used by Hammond[3]

$$\begin{aligned} \dot{y} &= \dot{y}_T - \dot{y}_M \\ \ddot{y} &= a_{Tn} - a_{Mn} \\ T\frac{\mathrm{d}a_{Mn}}{\mathrm{d}t} &+ a_{Mn} = u \\ \theta\frac{\mathrm{d}a_{Tn}}{\mathrm{d}t} &+ a_{Tn} = w(t) \end{aligned} \tag{31}$$

where $w(t)$ is a stationary white noise with

$$E[w(t_1)\, w(t_2)] = S\delta(t_1 - t_2)$$

Equation (2) thus becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left| \begin{array}{c} y \\ \dot{y} \\ a_{Mn} \\ a_{Tn} \end{array} \right| = \left| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1/T & 0 \\ 0 & 0 & 0 & -1/\theta \end{array} \right| \left| \begin{array}{c} y \\ \dot{y} \\ a_{Mn} \\ a_{Tn} \end{array} \right| + \left| \begin{array}{c} 0 \\ 0 \\ 1/T \\ 0 \end{array} \right| u + \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 1/\theta \end{array} \right| W \tag{32}$$

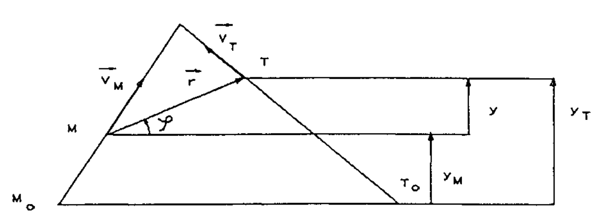

Fig. 1 Intercept geometry.

The last term of Eq. (32) is not considered because the vector state x is replaced by the estimated vector state $\hat{x}$. Thus, we have

$$\hat{x} = \begin{vmatrix} y \\ \dot{y} \\ a_{Mn} \\ a_{Tn} \end{vmatrix}; \quad F = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1/T & 0 \\ 0 & 0 & 0 & -1/\theta \end{vmatrix}; \quad G = \begin{vmatrix} 0 \\ 0 \\ 1/T \\ 0 \end{vmatrix} \tag{33}$$

The differential equation (22) for D gives

$$\begin{vmatrix} \dot{d}_1 \\ \dot{d}_2 \\ \dot{d}_4 \\ \dot{d}_4 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & -1/T & 0 \\ 0 & 1 & 0 & -1/\theta \end{vmatrix} \begin{vmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{vmatrix} \quad D_{tf}^T = [1,0,0,0] \tag{34}$$

The solution for $D(\tau)$ in this case may be written from Eq. (34) as

$$D^T = [1,\ \tau,\ T^2(1 - e^{-\tau/T}) - T\tau,\ \theta^2(e^{-\tau/\theta} - 1) + \theta\tau] \tag{35}$$

where τ is the time to go defined as

$$\tau = t_f - t \tag{36}$$

Using Eqs. (19) and (35), we may express P as

$$P = \begin{vmatrix} 1 & \tau & P_{13} & P_{14} \\ \tau & \tau^2 & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{31}^2 & P_{41}P_{31} \\ P_{41} & P_{42} & P_{41}P_{31} & P_{41}^2 \end{vmatrix} \tag{37}$$

where

$$\begin{aligned} P_{13} = P_{31} &= T^2(1 - e^{-\tau/T}) - T\tau \\ P_{23} = P_{32} &= \tau[T^2(1 - e^{-\tau/T}) - T\tau] \\ P_{14} = P_{41} &= \theta^2(e^{-\tau/\theta} - 1) + \theta\tau \\ P_{24} = P_{42} &= \tau[\theta^2(e^{-\tau/\theta} - 1) + \theta\tau] \end{aligned} \tag{38}$$

Using Eqs. (27) and (36) and

$$a = 1; \quad r = 3; \quad g_3 = \frac{1}{T}; \quad d_3 = T^2(1 - e^{-\tau/T}) - T\tau$$

we have

$$f(\tau) = \frac{1}{C_1} - \frac{1}{T^2}\int_\tau^0 [T^2(1 - e^{-\tau/T}) - T\tau]^2\, d\tau \tag{39}$$

or

$$f(\mu) = \frac{1}{C_1}\left[1 + C_1T^3\left(\mu - \mu^2 + \frac{\mu^3}{3} - 2\mu e^{-\mu} + \frac{1 - e^{-2\mu}}{2}\right)\right] \tag{40}$$

where

$$\mu = \frac{\tau}{T} \tag{41}$$

The optimal control law from Eq. (29) is then

$$u = -\frac{1}{T^2}C(\mu)\{y + \tau\dot{y} + [T^2(1 - e^{-\tau/T}) - T\tau]a_{Mn} + [\theta^2(e^{-\tau/\theta} - 1) + \theta\tau]a_{Tn}\} \tag{42}$$

where

$$C(\mu) = \frac{1 - e^{-\mu} - \mu}{(1/C_1T^3) + \mu - \mu^2 + \tfrac{1}{3}\mu^3 - 2\mu e^{-\mu} + \tfrac{1}{2}(1 - e^{-2\mu})} \tag{43}$$

and the predicted miss distance becomes

$$M = y + \tau\dot{y} + [T^2(1 - e^{-\tau/T}) - T\tau]a_{Mn} + [\theta^2(e^{-\tau/\theta} - 1) + \theta\tau]a_{Tn} \tag{44}$$

Conclusions

This Note has presented the analytic derivation of the optimal closed-loop guidance law for a finite-bandwidth homing missile intercepting a maneuvering target and has provided a simple analytic means of solving the matrix Riccati differential equation. The resulting optimal guidance law agrees with the result determined by Hammond and includes the result obtained by Cottrell and Asher. The major contribution of this Note lies in the analytic solution of the matrix Riccati differential equation.

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Estimation of Modal Parameters of Linear Structural Systems Using Hopfield Neural Networks

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I. Introduction

ARTIFICIAL neural networks (ANNs) can be considered highly interconnected dynamic systems wherein the main building block is a neuron. Two major types of artificial neural network architectures being investigated are feedforward- and recurrent-type network architectures. Previous research in feedforward-type ANNs show advantages in this type of network's ability to learn a control function.[1-4] Networks can be trained by a teacher that may be implementing a linear or nonlinear control algorithm. In essence, a feedforward-type neural network can be trained to approximate a hypersurface between the inputs and outputs. The use of ANNs for estimation purposes is rarely studied. This may be due to the fact that system dynamics cannot be explicitly retrieved from feedforward network connection strengths because the network learns only mapping functions. Recurrent-type neural network architectures do not suffer from this drawback. Recurrent-

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